1 Ellenőrző kérdések

1. Definiálja a primitív függvényt.
2. Mi a határozatlan integrál?
3. Mit nevezünk integrandusnak?
4. Írja le az alapintegrálokat!
5. Hogyan integrálunk összeget/különbséget?
6. Adja meg a parciális integrálás szabályát.
7. Adja meg a helyettesítéses integrálás szabályát!

2 Példák

1. Adjuk meg a következő határozatlan integrálokat:

$$a)\ \int 5\,dx \qquad b)\ \int \frac{3\sqrt[4]{x}}{x}\,dx \qquad c)\ \int \frac{x-2}{x+3}\,dx \qquad d)\ \int x^2\cdot e^x\,dx \qquad e)\ \int \sin^2 x\,dx$$

$$f)\ \int \cos(4x-1)\,dx \qquad g)\ \int \frac{\ln^2 x}{x}\,dx \qquad h)\ \int \frac{x}{\sqrt{1+3x}}\,dx$$

Megoldások:

$$\text{Alapintegrálok: } a)\ \int 5\,dx = 5x + c$$

$$b)\ \int \frac{3\sqrt[4]{x}}{x^3}\,dx = 3\int x^{-\frac{11}{4}}\,dx = 3\frac{x^{-\frac{7}{4}}}{-\frac{7}{4}} + c = -\frac{12}{7}\cdot x^{-\frac{7}{4}} + c = -\frac{12}{7}\cdot\frac{1}{\sqrt[4]{x^7}} + c$$

$$c)\ \int \frac{x-2}{x+3}\,dx = \int \frac{(x+3)-5}{x+3}\,dx = \int 1 - \frac{5}{x+3}\,dx = x - 5\cdot\ln|x+3| + c$$

$$\text{Parciális integrálás: } d)\ \int x^2\cdot e^x\,dx = x^2\cdot e^x - \int 2x\cdot e^x\,dx = x^2\cdot e^x - 2\left(x\cdot e^x - \int 1\cdot e^x\,dx\right) =$$

$$= x^2\cdot e^x - 2\left(x\cdot e^x - e^x\right) + c = x^2\cdot e^x - 2x\cdot e^x + 2e^x + c$$

$$e)\ \underline{\int \sin^2 x\,dx} = \int \sin x\cdot\sin x\,dx = \sin x\cdot(-\cos x) - \int \cos x\cdot(-\cos x)\,dx =$$

$$= -\sin x\cdot\cos x + \int \cos^2 x\,dx = -\sin x\cdot\cos x + \int \left(1-\sin^2 x\right)dx = \underline{-\sin x\cdot\cos x + x - \int \sin^2 x\,dx}$$

$$\text{Az aláhúzott egyenletet rendezve: } \int \sin^2 x\,dx = \frac{1}{2}\left(-\sin x\cdot\cos x + x\right) + c$$

Helyettesítéses integrálás: $f)\ \int \cos(4x-1)\,dx = \frac{1}{4}\int \cos(4x-1)\cdot 4\,dx = \frac{1}{4}\sin(4x-1)+c$

$g)\ \int \frac{\ln^2 x}{x}\,dx = \frac{1}{3}\int 3\ln^2 x\cdot\frac{1}{x}dx = \frac{1}{3}\ln^3 x + c$

$h)\ \int \frac{x}{\sqrt{1+3x}}\,dx$ Legyen: $\mathrm{t} = \sqrt{1+3\mathrm{x}} = (1+3x)^{\frac{1}{2}}$. Ekkor: $\frac{\mathrm{dt}}{\mathrm{dx}} = \frac{1}{2}(1+3x)^{-\frac{1}{2}}\cdot 3$, vagyis

$dt = \frac{3}{2}(1+3x)^{-\frac{1}{2}}dx.$ Valamint: $t^2 = 1+3x$, vagyis: $x = \frac{t^2-1}{3}.$

Tehát: $\int \frac{x}{\sqrt{1+3x}}\,dx = \frac{2}{3}\int x\cdot\frac{3}{2}(1+3x)^{-\frac{1}{2}}\,dx = \frac{2}{3}\int\frac{t^2-1}{3}dt = \frac{2}{9}\int t^2\,dt - \frac{2}{9}\int 1dt =$

$= \frac{2}{9}\frac{t^3}{3} - \frac{2}{9}t + c = \frac{2t^3}{27} - \frac{2}{9}t + c = \frac{2\left(\sqrt{1+3x}\right)^3}{27} - \frac{2}{9}(1+3x)+c$

3 Gyakorló feladatok

3.1 Integrálás megoldásokkal

1. Végezze el a következő integrálási feladatokat!

a) $\int\left(2e^x - 2^x + \frac{1}{2x}\right)dx =$ $\left(= 2e^x - \frac{2^x}{\ln 2} + \frac{1}{2}\cdot\ln|x| + c\right)$

b) $\int\left(\frac{\cos x}{\pi} - \frac{3}{\mathrm{ch}^2 x} + \frac{1}{\sqrt{4+4x^2}}\right)dx =$ $\left(= \frac{\sin x}{\pi} - 3\,\mathrm{th}\,x + \frac{1}{2}\cdot\mathrm{arsh}\,x + c\right)$

c) $\int\left(1+\sqrt{x}\right)^4 dx =$ $\left(= x + \frac{8}{3}x\cdot\sqrt{x} + 3x^2 + \frac{8}{5}x^2\cdot\sqrt{x} + \frac{x^3}{3} + c\right)$

d) $\int\left(\sqrt{x}+1\right)\cdot\left(x-\sqrt{x}+1\right)dx =$ $\left(= \frac{2}{5}x^2\cdot\sqrt{x} + x + c\right)$

e) $\int\frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\,dx =$ $\left(= \frac{6}{7}x\cdot\sqrt[6]{x} - \frac{1}{\sqrt[4]{x}} + c\right)$

f) $\int\frac{\sqrt{x^3}+1}{\sqrt{x}+1}\,dx =$ $\left(= \frac{x^2}{2} - \frac{2}{3}x\cdot\sqrt{x} + x + c\right)$

g) $\int\frac{1+2x^2}{x^2\cdot\left(1+x^2\right)}\,dx =$ $\left(= ar\mathrm{ctg}\,x - \frac{1}{x} + c\right)$

h) $\int\frac{\sqrt{9x^2-9}-1}{\sqrt{x^2-1}}\,dx =$ $\left(= 3x - \mathrm{arch}\,x + c\right)$

i) $\int\frac{1+\cos^2 x}{1+\cos 2x}\,dx =$ $\left(= \frac{1}{2}\cdot\left(x + \mathrm{tg}\,x\right) + c\right)$

j) $\int \frac{x+\cos^2 x}{x\cdot\cos^2 x}\,dx=$ $\left(=\operatorname{tg} x+\ln|x|+c\right)$

k) $\int x^3\cdot\sin x\,dx=$ $\left(=\left(6x-x^3\right)\cdot\cos x+\left(3x^2-6\right)\cdot\sin x+c\right)$

l) $\int x\cdot 5^x\,dx=$ $\left(=\frac{5^x}{\ln 5}\cdot\left(x-\frac{1}{\ln 5}\right)+c\right)$

m) $\int\left(x^2+3x-1\right)\cdot e^x dx=$ $\left(=\left(x^2+x-2\right)\cdot e^x+c\right)$

n) $\int (x-1)^3\cdot\log_2 x\,dx=$

$\left(=\frac{(x-1)^4}{4}\cdot\log_2 x-\frac{1}{\ln 2}\cdot\left(\frac{x^4}{4}-\frac{4}{3}x^3+3x^2-4x+\ln|x|\right)+c\right)$

o) $\int\ln\left(x^2+1\right)\,dx=$ $\left(=x\cdot\ln\left(x^2+1\right)-2\cdot(x-\operatorname{arctg} x)+c\right)$

p) $\int x\cdot\operatorname{arctg} x\,dx=$ $\left(=\frac{1}{2}\cdot\left(x^2\cdot\operatorname{arctg} x+\operatorname{arctg} x-x\right)+c\right)$

q) $\int e^x\cdot\sin x\,dx=$ $\left(=\frac{e^x}{2}\cdot(\sin x-\cos x)+c\right)$

r) $\int\sin x\cdot\operatorname{ch} x\,dx=$ $\left(=\frac{1}{2}\cdot(\sin x\cdot\operatorname{sh} x-\cos x\cdot\operatorname{ch} x)+c\right)$

s) $\int(5x-4)^6\,dx=$ $\left(=\frac{1}{35}\cdot(5x-4)^7+c\right)$

t) $\int\cos(7x+2)\,dx=$ $\left(=\frac{1}{7}\cdot\sin(7x+2)+c\right)$

u) $\int\frac{1}{\operatorname{ch}^2(3x+1)}\,dx=$ $\left(=\frac{1}{3}\cdot\operatorname{th}(3x+1)+c\right)$

v) $\int\frac{\operatorname{th}^5 x}{\operatorname{ch}^2 x}\,dx=$ $\left(=\frac{1}{6}\cdot\operatorname{th}^6 x+c\right)$

w) $\int\frac{x}{\sqrt[5]{x^2-1}}\,dx=$ $\left(=\frac{5}{8}\cdot\sqrt[5]{\left(x^2-1\right)^4}+c\right)$

x) $\int(x+1)\cdot\sqrt{x^2+2x-3}\,dx=$ $\left(=\frac{1}{3}\cdot\sqrt{\left(x^2+2x-3\right)^3}+c\right)$

y) $\int\frac{\ln^2 x}{x}\,dx=$ $\left(=\frac{1}{3}\cdot\ln^3 x+c\right)$

z) $\int\frac{2x-1}{x^2-x+9}\,dx=$ $\left(=\ln\left|x^2-x+9\right|+c\right)$

aa) $\int\frac{x^2}{x^3+1}\,dx=$ $\left(=\frac{1}{3}\cdot\ln\left|x^3+1\right|+c\right)$

bb) $\int\frac{e^x+1}{e^x+x}\,dx=$ $\left(=\ln\left|e^x+x\right|+c\right)$

cc) $\int \operatorname{tg} x\, dx =$ $\quad \left(= -\ln\left|\cos x\right| + c\right)$

dd) $\int \frac{1}{x \cdot \ln x}\, dx =$ $\quad \left(= \ln\left|\ln x\right| + c\right)$

ee) $\int x \cdot e^{-x^2}\, dx =$ $\quad \left(= -\frac{1}{2} \cdot e^{-x^2} + c\right)$

ff) $\int \frac{\operatorname{ctg}^3 x}{\sin^2 x}\, dx =$ $\quad \left(= -\frac{1}{4} \cdot \operatorname{ctg}^4 x + c\right)$

gg) $\int 2^{\operatorname{ch} x} \cdot \operatorname{sh} x\, dx =$ $\quad \left(= \frac{2^{\operatorname{ch} x}}{\ln 2} + c\right)$

hh) $\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\, dx =$ $\quad \left(= \frac{1}{3} \cdot \arcsin^3 x + c\right)$

ii) $\int \sqrt{4x^2 + 4x + 5}\, dx =$ $\quad \left(= \frac{2x+1}{4} \cdot \sqrt{4x^2 + 4x + 5} + \operatorname{arsh} \frac{2x+1}{2} + c\right)$

jj) $\int \sqrt{8 - 2x - x^2}\, dx =$ $\quad \left(= (x+1) \cdot \sqrt{8 - 2x - x^2} + 9 \cdot \arcsin \frac{x+1}{3} + c\right)$

kk) $\int \sqrt{x^2 - 2x - 1}\, dx =$ $\quad \left(= \frac{x-1}{\sqrt{2}} \cdot \sqrt{x^2 - 2x - 1} - \operatorname{arch} \frac{x-1}{\sqrt{2}} + c\right)$

ll) $\int \frac{x^2}{\sqrt{1 - 2x - x^2}}\, dx =$ $\quad \left(= \frac{x+5}{2} \cdot \sqrt{1 - 2x - x^2} + 2 \cdot \arcsin \frac{x+1}{\sqrt{2}} + c\right)$

mm) $\int \frac{x^3}{x+1}\, dx =$ $\quad \left(= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln\left|x\right| + c\right)$

nn) $\int \frac{1}{x^2 - 2x - 3}\, dx =$ $\quad \left(= \frac{1}{4} \cdot \ln\left|\frac{x-3}{x+1}\right| + c\right)$

oo) $\int \frac{x^2 - 1}{(x+2)^3}\, dx =$ $\quad \left(= -\frac{3}{2} \cdot \frac{1}{(x+2)^2} + \frac{4}{x+2} + \ln\left|x+2\right| + c\right)$

pp) $\int \frac{3x+1}{4x^2 - 12x + 9}\, dx =$ $\quad \left(= -\frac{11}{4} \cdot \frac{1}{2x-3} + \frac{3}{4} \cdot \ln\left|2x-3\right| + c\right)$

qq) $\int \frac{x^3 - 1}{4x^3 - x}\, dx =$ $\quad \left(= \frac{x}{4} + \ln\left|x\right| - \frac{7}{16} \cdot \ln\left|2x-1\right| - \frac{9}{16} \cdot \ln\left|2x+1\right| + c\right)$

rr) $\int \frac{x}{x^4 - 3x^2 + 2}\, dx =$ $\quad \left(= \frac{1}{2} \cdot \ln\left|\frac{x^2 - 2}{x^2 - 1}\right| + c\right)$

ss) $\int \frac{\sqrt{x}}{x \cdot \sqrt{x} - 1}\, dx =$ $\quad \left(= \frac{2}{3} \cdot \ln\left|\sqrt{x^3} - 1\right| + c\right)$

tt) $\int \frac{\sqrt{x}}{\sqrt[3]{x}-1}\,dx =$ $\left(= \frac{x}{6} + \frac{\sqrt[3]{x^2}}{4} + \frac{\sqrt[3]{x}}{2} + \frac{1}{2}\cdot \ln\left|\sqrt{x^3}-1\right| + c\right)$

uu) $\int \frac{1-e^{-x}}{e^{2x}-1}\,dx =$ $\left(= \ln\left|\frac{e^x+1}{e^x}\right| - \frac{1}{e^x} + c\right)$

vv) $\int \frac{2^{3x}}{2^{2x}-1}\,dx =$ $\left(= \begin{cases} \frac{1}{\ln 2}\cdot\left(2^x - \operatorname{arth} 2^x\right) + c, & \text{ha } x<0 \\ \frac{1}{\ln 2}\cdot\left(2^x - ar\operatorname{cth} 2^x\right) + c, & \text{ha } x>0 \end{cases}\right)$

ww) $\int \frac{\cos x}{2-\cos x}\,dx =$ $\left(= \frac{1}{1-\operatorname{tg}\frac{x}{2}} + \frac{1}{2}\cdot\ln\left|1+\operatorname{tg}^2\frac{x}{2}\right| + c\right)$

xx) $\int \frac{1}{\sin x\cdot(\cos x+2)}\,dx =$ $\left(= \operatorname{tg}\frac{x}{2} - \frac{2\sqrt{3}}{3}\cdot ar\operatorname{ctg}\frac{ar\operatorname{ctg}\frac{x}{2}}{\sqrt{3}} + c\right)$

3.2 Alapintegrálok

1. Számítsa ki a következő integrálokat!

a) $\int \frac{x+1}{\sqrt{x}} + 2\cos x - \frac{3}{\cos^2 x} + \frac{5}{x} + \frac{6}{\sqrt[3]{x^2}} + 2dx$

b) $\int \operatorname{tg}^2 x + \frac{\cos^2 x - \cos 2x}{2\cos^2 x}\,dx$

c) $\int \frac{5\cos 2x}{\sin x + \cos x}dx$

d) $\int \frac{\sqrt[3]{x^2} - 4x^5 + x^6}{x^7} + 3\sin x + \frac{9}{\sin^2 x} + 5dx$

3.3 Helyettesítéses integrálás

1. A helyettesítéses integrálás módszerével határozza meg az alábbi integrálokat!

a) $\int 5\cos(4x+7) - \frac{5}{\sqrt[3]{9x+6}}\,dx$

b) $\int \frac{3}{7x+6} + \frac{1}{(4x-9)^5}\,dx$

c) $\int \frac{3}{\cos^2(7x+6)} - \frac{2}{4x+3} + 6 \cdot e^{5x+6} + 5^{4x-9}\, dx$

3.3 Integrálás gyakorlása

1. Határozza meg az alábbi integrálokat!

a) $\int (\sin x)^2\, dx$

b) $\int (3x+5) \cdot \sin x dx$

c) $\int e^x \cdot (5x+8) dx$

d) $\int x \cdot e^x dx$

e) $\int \left(3x^2+3\right) \sin x dx$

f) $\int x^2 \cdot \ln x dx$

g) $\int \ln x dx$

h) $\int 2e^x \cdot \cos x dx$

i) $\int 5 \sin x \cdot \cos x dx$